

Chapter 9 : Doppler Ambiguity Function

I. Limitations of Doppler Ultrasound

- In principle, aliasing only occurs in PW and Color Doppler, in which signals are effectively sampled at an interval of PRI. In a digital system, however, aliasing is also present in CW. This is because the sampling frequency of the digitizer limits the maximum frequency that can be detected without aliasing based on the Nyquist criterion. Therefore, the maximum un-aliased velocity is also limited. Generally, the sampling frequency of the digitizer is chosen such that the maximum detectable velocity is higher than that of most physiological flows.
- In PW Doppler and Color Doppler, the maximum velocity (v_{\max}) that can be detected without aliasing is

$$v_{\max} \leq \frac{\lambda}{4 \cdot PRI},$$

where λ is the wavelength. On the other hand, the maximum range of the Doppler gate for a given PRI is

$$R_{\max} \leq \frac{c \cdot PRI}{2},$$

where c is the sound velocity in the body. Therefore,

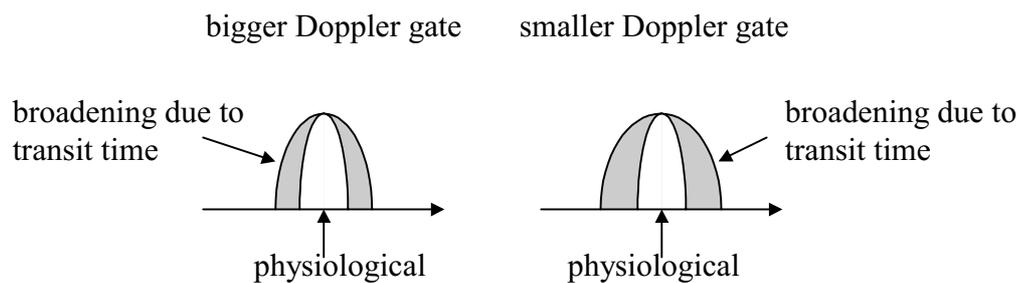
$$v_{\max} R_{\max} \leq \frac{c\lambda}{8}.$$

In other words, there exists a tradeoff between the maximum velocity that can be detected without aliasing and the maximum depth of the Doppler gate. In practice, a lower Doppler frequency is often preferred since it allows detection of higher blood velocities and deeper flows. In addition, lower frequencies have lower attenuation and hence they can penetrate deeper (assuming the increase in backscattered signal does not exceed the increase in attenuation).

- Blood flows detected by Doppler instruments always appear as a range of frequencies, instead of a single frequency. One of the main reasons is that blood moves at a variety of velocities throughout the vessel. Therefore, flows within a sample volume have various Doppler shifts and the spectral width is broadened. The amount of spectral broadening depends on the nature of the blood flow and

can vary under different physiological conditions.

- There is a physical limitation in all Doppler measurements which also causes spectral broadening. This limitation is due to the transit time required for red blood cells to propagate through the Doppler gate. In other words, since the blood cells are moving quickly through the Doppler gate, the Doppler instrument only has a certain amount of time to estimate the Doppler shifts corresponding to the moving blood cells. To increase the frequency resolution, the transit time needs to be increased and this decreases the time (range) resolution. This uncertainty relation is a well known physical property (Heisenberg's Uncertainty Principle) which states that the position and the velocity cannot be accurately estimated simultaneously. Since the Doppler gate is always limited, this type of spectral broadening is inevitable. Note that the size of the Doppler gate is also limited in CW, since the Doppler gate can be defined as the intersection of the CW beam and the vessel.
- Since the transit time is determined by the size of the Doppler gate and the blood velocity, it is obvious that the amount of spectral broadening increases for flows with higher velocities (i.e., shorter transit times).
- Consider two Doppler gates with different sizes, the effects of spectral broadening can be illustrated by the following two spectra:



- To reduce the spectral broadening resulting from the transit time, the size of the Doppler gate (i.e., the pulse length and the beam width) may be increased to reduce the spectral width such that errors of the maximum and minimum velocity measurements can be decreased.

II. Matched Filtering

- Based on the uncertainty principle, an important character of Doppler measurement is that the range and the Doppler shift cannot be measured accurately at the same time. In this section, we will show that the accuracy can be improved with signal-to-noise ratio (SNR) in both cases, although the fundamental tradeoff between these two resolutions still exists. Specifically, we will introduce a linear filter that maximizes the SNR. This filter is generally known as the matched filter.
- For a given input signal $x(t)$ and a filter $h(t)$, the filter output $y(t)$ can be found by the following convolution integral in the time domain or the following multiplication in the frequency domain

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

$$Y(f) = X(f)H(f)$$

Note that $X(f)$, $H(f)$ and $Y(f)$ are the Fourier transforms of $x(t)$, $h(t)$ and $y(t)$, respectively.

- Define the SNR as the following

$$SNR \equiv \frac{\text{peak instantaneous output signal power}}{\text{output noise power}},$$

the goal is to determine the optimal form of the filter $h(t)$, such that the SNR is maximized.

- Suppose the filter output $y(t)$ is maximized at the time $t = T_d$, we have

$$y(T_d) = \int_{-\infty}^{\infty} x(T_d - \tau) h(\tau) d\tau.$$

If we assume the noise input to the filter (denoted by $X_N(f)$) is uncorrelated with the filter and has frequency independent distribution as a function of frequency (i.e., $\langle |X_N(f)|^2 \rangle \equiv N_0$), the total noise output power (σ^2) becomes

$$\sigma^2 = \left\langle \int_{-\infty}^{\infty} |X_N(f)H(f)|^2 df \right\rangle = N_0 \int_{-\infty}^{\infty} |H(f)|^2 df.$$

Using the Parseval's theorem, we have

$$\sigma^2 = N_0 \int_{-\infty}^{\infty} h(t)^2 dt.$$

- The goal of matched filtering is to maximize $y^2(T_d)/\sigma^2$. By using the Schwarz's inequality, we have

$$SNR = \frac{\left| \int_{-\infty}^{\infty} x(T_d - \tau)h(\tau) d\tau \right|^2}{N_0 \int_{-\infty}^{\infty} h^2(t) dt} \leq \frac{\int_{-\infty}^{\infty} x^2(T_d - t) dt \int_{-\infty}^{\infty} h^2(t) dt}{N_0 \int_{-\infty}^{\infty} h^2(t) dt}.$$

The SNR is maximized if

$$h(t) = \alpha x(T_d - t),$$

where α is a constant.

- The filter given by the above equation is known as a matched filter. Its response is a time-delayed version of the time-reversed input signal and it gives a maximum SNR of

$$SNR_{\max} = \frac{\int_{-\infty}^{\infty} x^2(T_d - t) dt}{N_0} = \frac{\int_{-\infty}^{\infty} x^2(t) dt}{N_0} \equiv \frac{E}{N_0}.$$

In other words, the maximum signal-to-noise ratio is determined only by its total energy E , not by the detailed structure of the signal.

- For a complex signal, a matched filter can also be found using the same approach. The matched filter for complex signals generally has the following form

$$h(t) = \alpha x^*(T_d - t),$$

where $*$ represents the complex conjugate.

III. Range and Velocity Accuracy

- The received signal is usually corrupted by noise, and this decreases the accuracy of both the range and the frequency resolutions in Doppler ultrasound. It can be shown that the resolution depends directly on the SNR, and therefore the optimal resolution is obtained by using a matched filter.
- Let the total input $x(t)$ be the sum of a noise-free signal $x_0(t)$ and a noise signal $n(t)$, i.e.,

$$x(t) = x_0(t) + n(t).$$

Using a matched filter $h(t) = x_0(-t)$, the maximum output occurs at $t = 0$ and

$$y(t) = \int_{-\infty}^{\infty} x_0(\tau)x_0(\tau-t)d\tau + \int_{-\infty}^{\infty} n(\tau)x_0(\tau-t)d\tau \equiv y_0(t) + \int_{-\infty}^{\infty} n(\tau)x_0(\tau-t)d\tau.$$

Keep in mind that the noise-free output $y_0(t)$ has its maximum at $y_0(0)$. Suppose the estimated value of the time when $y_0(t)$ is maximized has an error of Δt , our goal is to derive its mean square error, i.e., $\langle \Delta t^2 \rangle$.

- Using a Taylor expansion, we have

$$y_0(\Delta t) = y_0(0) + \frac{\Delta t^2}{2} y_0''(0) + R,$$

where R is the remainder. Note that $y_0'(0) = 0$ since $y_0(t)$ has its maximum at $t = 0$. By defining $y_0(0) = \int_{-\infty}^{\infty} x_0^2(\tau)d\tau \equiv E$ (the total energy of the signal), the above equation can be re-written as (keeping only up to the second order term)

$$y_0^2(\Delta t) = y_0^2(0) + \Delta t^2 y_0''(0) E + R.$$

- $y_0''(0)$ is related to the frequency bandwidth of $x_0(t)$ and it can be shown as the following. Since

$$y_0''(t) = \int_{-\infty}^{\infty} x_0''(\tau-t)x_0(\tau)d\tau,$$

thus

$$y_0''(0) = \int_{-\infty}^{\infty} x_0''(\tau) X_0(\tau) d\tau = -4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f) df.$$

Defining a quantity β as follows:

$$\beta^2 \equiv \frac{4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f) df}{\int_{-\infty}^{\infty} X_0(f) X_0^*(f) df} = \frac{4\pi^2 \int_{-\infty}^{\infty} f^2 X_0(f) X_0^*(f) df}{E},$$

we have

$$y_0''(0) = -\beta^2 E$$

and

$$y_0^2(\Delta t) - y_0^2(0) = -\beta^2 E^2 \Delta t^2.$$

- Define a noise signal as the following

$$\langle \epsilon^2 \rangle \equiv \langle y_0^2(0) - y_0^2(\Delta t) \rangle \equiv \left\langle \int_{-\infty}^{\infty} |n(\tau) X_0(\tau)|^2 d\tau \right\rangle,$$

the following equation can be obtained by using the Parseval's theorem and by assuming that the noise is un-correlated with the signal. We have

$$\langle \epsilon^2 \rangle = N_0 E = \beta^2 E^2 \langle \Delta t^2 \rangle.$$

Therefore,

$$\langle \Delta t^2 \rangle = \frac{1}{\beta^2 (E/N_0)}$$

and it is clear that as the SNR improves (optimal SNR can be obtained with a matched filter), the time error decreases.

- It can be shown by methods very similar to the above derivation that the mean square error in the frequency measurement is

$$\langle \Delta f^2 \rangle = \frac{1}{\alpha^2 (E/N_0)},$$

where α is defined as

$$\alpha^2 \equiv \frac{4\pi^2 \int_{-\infty}^{\infty} t^2 x_0^2(t) dt}{\int_{-\infty}^{\infty} x_0^2(t) dt}.$$

Thus we find that the accuracy of the frequency measurement can also be optimized by matched filtering.

- Combining the equation for both the time and the frequency measurement errors, we have

$$\left[\langle \Delta t^2 \rangle \langle \Delta f^2 \rangle \right]^{1/2} = \frac{1}{\alpha\beta (E/N_0)}.$$

From the above equation, we find that the accuracy of both the time and the frequency measurements can be improved simultaneously by improving the SNR and by increasing the time-bandwidth product (i.e., $\alpha\beta$). For a given system, however, the fundamental tradeoff between those two still remains.

- By using Schwarz's inequality, it can be found that $\alpha\beta$ is always larger than a constant, and the minimum occurs when $Atx_0(t) = -x'_0(t)$ with A being a constant. Since a Gaussian signal satisfies the above relation, we find that Gaussian signals give poorer simultaneous measurements of time and frequency than any other signal.

IV. Doppler Ambiguity Function

- For a given input signal, we have shown that matched filtering provides optimal range and Doppler shift resolution. Another important task when designing a Doppler instrument is, based on specific requirements, to choose a waveform for the least amount of ambiguity after the matched filtering process. To do this, it is convenient to define the following function

$$\chi(\Delta t, \Delta f) = \int_{-\infty}^{\infty} u(\tau) u^*(\tau - \Delta t) e^{j2\pi\tau\Delta f} d\tau,$$

where $u(\tau)$ is the analytic signal of a real signal $x(\tau)$ and $x(\tau) = (u(\tau) + u^*(\tau))/2$.

- The ambiguity function is defined as $|\chi(\Delta t, \Delta f)|^2$ and its utility can be seen by examining its properties. Let $\Delta f = 0$,

$$|\chi(\Delta t, 0)|^2 = \left| \int_{-\infty}^{\infty} u(\tau) u^*(\tau - \Delta t) d\tau \right|^2.$$

The ambiguity function in this case is simply the squared magnitude of the autocorrelation function of the received signal. Obviously the ambiguity function along this axis describes the axial (range) resolution, since the range resolution is determined by the temporal duration of the received signal.

- Similarly, we can examine the ambiguity function along the Δf axis by setting $\Delta t = 0$. We have

$$|\chi(0, \Delta f)|^2 = \left| \int_{-\infty}^{\infty} u(\tau) u^*(\tau) e^{j2\pi\tau\Delta f} d\tau \right|^2 = \left| \int_{-\infty}^{\infty} |u(\tau)|^2 e^{j2\pi\tau\Delta f} d\tau \right|^2.$$

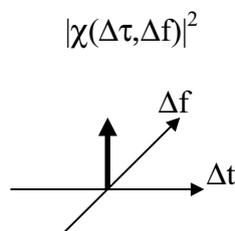
Therefore, the ambiguity function along the Δf axis is simply the power spectrum of the squared envelope of the received signal. Clearly the broader the bandwidth of the signal, the more difficult it is to resolve the presence of two Doppler shifts from objects with different velocities.

- Using the Parseval's theorem, the following relation can be obtained

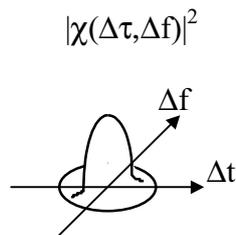
$$\int \int_{-\infty}^{\infty} |\chi(\Delta t, \Delta f)|^2 d(\Delta t) d(\Delta f) = |\chi(0, 0)|^2.$$

In other words, the total potential ambiguity is the same for all signals that possess the same energy (which is proportional to $|\chi(0, 0)|$). This property is closely related to the uncertainty principles that we previously derived. The goal of designing a Doppler instrument is to distribute the ambiguity in an optimal way based on particular requirements.

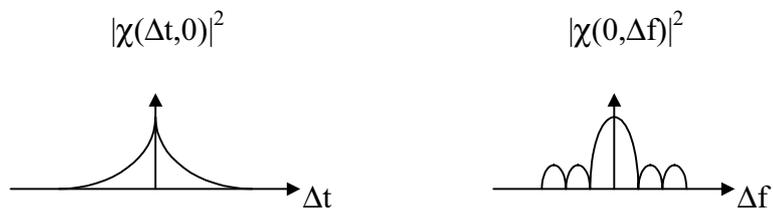
- Ideally, the ambiguity function should consist of a single peak with infinitesimal width at the origin and zero elsewhere, i.e.,



Obviously, such a function is impossible to achieve. A reasonable approximation of the ideal ambiguity function is shown below. In this case, there is no ambiguity since there is only one peak. The accuracy, however, is still determined by the width of the peak and may or may not satisfy the design requirements.



- The ambiguity function for a simple rectangular pulse appears as the square of a triangular function along the Δt axis and as the square of a sinc function along the Δf axis. This is conceptually shown in the following.



- Since the total volume under the ambiguity function is fixed for a given input energy, the width of peak at the origin may be decreased at the price of peaks being formed at regions other than the peak at the origin. These peaks give rise to ambiguities. Therefore, the requirement for accuracy and ambiguity may not always be possible to satisfy at the same time, i.e., another fundamental tradeoff exists in Doppler ultrasound.