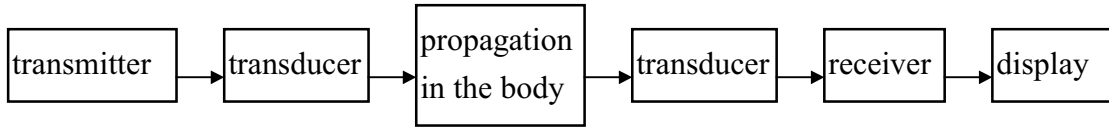


Chapter 5 : Diffraction and Beam Formation Using arrays

I. Imaging Model

- Ultrasonic image formation can be described by the following model



Considering one dimensional situations (A-scan), the received signal $V(t)$ is given by

$$V(t) = k \iiint \frac{R(x', y', z') e^{-2\beta z'}}{z'} B(x', y', z') p(t - \frac{2z'}{c}) dx' dy' dz',$$

where $R(x', y', z')$ is the reflectivity of the body at arbitrary position (x', y', z') , $B(x', y', z')$ is the pulse-echo radiation pattern and $p(t - 2z'/c)$ is the received pulse-echo signal from an ideal reflector at depth z' .

By scanning in the x direction and assuming the attenuation $(e^{-2\beta z'})$ and the spreading term $(1/z')$ are corrected, the following equation yields

$$S(x, t) = k \iiint R(x', y', z') B(x' - x, y', z') p(t - \frac{2z'}{c}) dx' dy' dz'.$$

Note that without loss of generality, $B(x' - x, y', z')$ can be re-written as $B(x - x', y', z')$ (a simple sign change) such that the above equation can be viewed as a normal convolution in x .

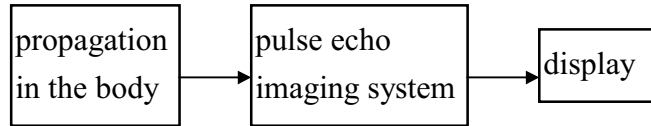
- In general, the pulse-echo waveform can be described by

$$p(t - \frac{2z'}{c}) = A(t - \frac{2z'}{c}) \cos(2\pi f_0 (t - \frac{2z'}{c})),$$

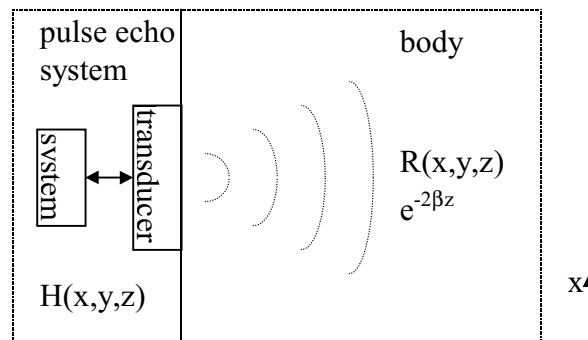
where $A(\cdot)$ is the envelope of the pulse and f_0 is the carrier frequency of the

pulse and the waveform distortion due to frequency dependent attenuation is ignored in the above equation.

- The ultimate goal of imaging (assuming 2D) is to have $S(x, t) = R(x, y_0, ct/2)$, where y_0 is the position of the image plane. In other words, $B(x - x', y', z')$ needs to be $\delta(x - x', y' - y_0)$ and $A(t - 2z'/c)$ needs to be $\delta(t - 2z'/c)$. In practice, however, both $A(\cdot)$ and $B(\cdot)$ have a finite extent. Therefore, the size of a pixel in an ultrasonic system is determined by the point spread function, which is determined by the beam width of $B(\cdot)$ and the pulse width of $A(\cdot)$.
- $B(\cdot)$ is mainly determined by diffraction and will be discussed in detail in the following section. Assuming an impulse excitation (i.e., flat frequency spectrum), $A(\cdot)$ is then determined by the transducer impulse response. Typically, a transducer has a fractional bandwidth (bandwidth divided by the center frequency) of about 70-100%. Such bandwidth results in a two-way duration of about 2-3 periods of the carrier. In other words, the resolution in depth is about 2-3 wavelengths.
- Assuming a linear system, the aforementioned model can then be simplified as below



In other words, the received signal can be obtained by convolving the impulse response of the imaging system with the distribution of reflectivity of the body (including attenuation). This model is also applicable for full three-dimensional scanning situations. Hence,



$$t=2z/c \quad \xrightarrow{\quad z \quad}$$

i.e.,

$$S(x, y, z) = H(x, y, z) *** R(x, y, z) \Big|_{z=ct/2}.$$

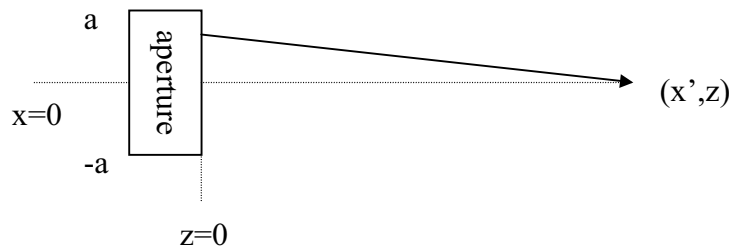
II. Diffraction from One-Dimensional Apertures

- As in optics, diffraction from a two-dimensional aperture determines the resolution in a plane ($x-y$ plane) perpendicular to the direction of wave propagation (z direction). For two-dimensional ultrasonic imaging and for the sake of simplicity, we will only consider one-dimensional apertures for our analysis. In other words, we will only be interested in the x direction and ignore the y direction.
- Assuming every point along an aperture can be modeled as an isotropic radiator of sound, the pressure wave at carrier frequency f_0 is (i.e., the free-space Green's function)

$$p(R) = A_0 \frac{e^{jkR}}{R},$$

where R is the distance between the radiator and the observer, A_0 is a constant which will be ignored, and k is the wave number ($k = 2\pi / \lambda$, λ is the wavelength).

- The continuous wave (CW) radiation pattern from a finite aperture is simply a superposition of all point sources. Therefore, for a point at (x', z) , the pressure wave becomes



$$p(x', z) = \int_{-a}^a \frac{e^{jkd(x, x')}}{d(x, x')} dx,$$

where $d(x, x')$ is the distance from a point x in the aperture plane to a point x' in plane z and the aperture is non-zero between $(-a, a)$.

- In the Fresnel region, which is defined as $z^2 \gg (x - x')^2$,

$$d(x, x') = z \left(1 + \frac{(x - x')^2}{z^2} \right)^{1/2} \approx z + \frac{(x - x')^2}{2z}.$$

and the pressure wave can be approximated by

$$p(x', z) \approx \frac{1}{z} \int_{-a}^a e^{jkz} e^{jk(x - x')^2 / 2z} dx = \frac{e^{jkz} e^{jkx'^2 / 2z}}{z} \int_{-a}^a e^{-jkxx' / z} e^{jkx^2 / 2z} dx.$$

Note that with an array, Fresnel approximation can be guaranteed by using a constant f-number for aperture opening in the near field. The above equation can be generalized for the case of an arbitrary complex aperture function $C(x) = |C(x)|e^{j\theta(x)}$,

$$p(x', z) \approx \frac{e^{jkz} e^{jkx'^2 / 2z}}{z} \int_{-a}^a C(x) e^{-jkxx' / z} e^{jkx^2 / 2z} dx.$$

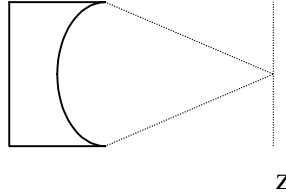
- In the far field, where $ka^2 / 2z \ll 1$, the integral reduces to

$$\begin{aligned} p(x', z) &\approx \frac{e^{jkz} e^{jkx'^2 / 2z}}{z} \int_{-a}^a C(x) e^{-jkxx' / z} dx \\ &= \frac{e^{jkz} e^{jkx'^2 / 2z}}{z} F.T.[C(x)] \end{aligned},$$

where F.T. stands for Fourier transform.

- When not in the far field, $C(x) e^{jkx^2 / 2z}$ can be viewed as the effective aperture function. Furthermore, if $C(x)$ is chosen to be $C(x) = |C(x)|e^{-jkx^2 / 2z}$, the integral reduces to the same equation as in the far field, and hence the Fourier transform relation still holds.

- An effective aperture $C(x) = |C(x)|e^{-jkx^2/2z}$ simply means a lens focused at a depth z , i.e.,



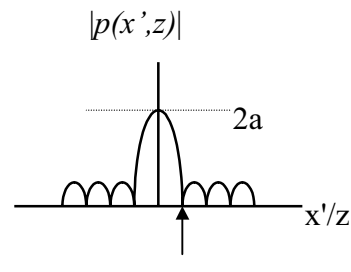
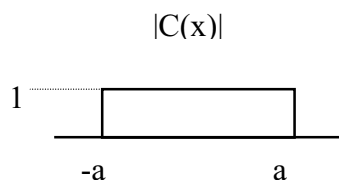
Letting z_0 be the fixed focal point of the lens, the effective aperture function at

an arbitrary range z becomes $C'(x) = |C(x)|e^{\frac{jkx^2}{2}\left(\frac{1}{z} - \frac{1}{z_0}\right)}$. In other words, the Fourier transform relation is still valid by using the new aperture function $C'(x)$. On the other hand, the Fourier transform of the radiation pattern (i.e., the scaled aperture function) can be viewed as the spatial frequency spectrum.

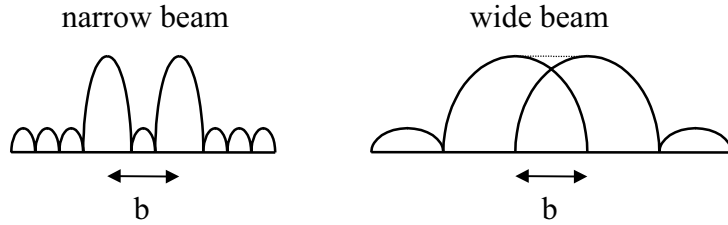
- For a uniformly weighted aperture, the radiation pattern (in the far field or in focus) is simply a sinc function in combination with some phase terms. In addition, since the width of the diffraction pattern is directly related to the lateral resolution (x' direction), it is clear that the aperture size ($2a$) is inversely proportional to the beam width and therefore proportional to the lateral resolution (i.e., the larger the aperture, the better the lateral resolution). Similarly, the higher the frequency is, the better the lateral resolution can be achieved. This is illustrated as follows.
- Assuming $C(x)$ is a rectangular aperture from $-a$ to a (the phase term can be ignored),

$$\begin{aligned} |p(x', z)| &= \left| \int_{-a}^a e^{-jkxx'/z} dx \right| = \left| \frac{1}{jkx'/z} \left[e^{jkx'a/z} - e^{-jkx'a/z} \right] \right| \\ &= \left| 2a \frac{\sin kx'a/z}{kx'a/z} \right| = \left| 2a \operatorname{sinc} \left(\frac{kx'a}{z} \right) \right| \end{aligned}$$

The first zero-crossing point occurs when $\frac{kx'a}{z} = \pi$, or when $\frac{x'}{z} = \frac{\lambda}{2a}$.



Considering two objects spaced b apart in the direction of x' , the lateral response of these two objects can be modeled as the convolution of the objects with the lateral beam pattern. It is clear that a narrower beam width produces better lateral resolution, as illustrated as below.



- In practice, -3dB or -6dB beam width is usually used to define lateral resolution.
- The above equations are derived for a transmitted wave, the diffraction analysis is exactly the same for reception due to the reciprocal nature. Therefore, assuming a fixed aperture, the two-way radiation pattern becomes

$$B(x', z) = T(x', z, \omega_0) R(x', z, \omega_0) = p(x', z)^2,$$

where $\omega_0 = 2\pi f_0$ and f_0 is the carrier frequency. In general, the transmit and receive apertures are different, thus resulting in different radiation patterns.

- The actual pulse-echo system uses a pulse (i.e., a broad band signal) rather than a continuous wave. Based on the principle of superposition, the broad band radiation pattern is

$$B(x', z) = \int T(x', z, \omega) R(x', z, \omega) A(\omega) d\omega,$$

where $A(\omega)$ is the spectrum of the pulse excitation.

III. Diffraction and Propagation Delays

- Rather than modeling continuous wave diffraction, we can also derive the radiation pattern using pulse propagation from a point in the aperture to a point in the field. In other words,

$$p(x', z, t) = \int_{-a}^a A(t - \tau(x, x', z)) \cos \omega_0(t - \tau(x, x', z)) dx,$$

where $\tau(x, x', z)$ is the propagation time from a point $(x, 0)$ to a point (x', z) and

$$\tau(x, x', z) = \left((x - x')^2 + z^2 \right)^{\frac{1}{2}} / c,$$

where c is the propagation velocity of sound waves. In the Fresnel region,

$$\tau(x, x', z) \approx \frac{z}{c} + \frac{(x - x')^2}{2zc}.$$

Note that the first term can be viewed as an overall propagation delay (setting $x = x' = 0$) and the second term is a parabolic term which varies with z .

- Similar to the continuous wave derivation for diffraction, we can apply propagation delays across the aperture as the following

$$\tau'(x, x', z) = \frac{x^2}{2zc},$$

therefore,

$$\begin{aligned} p(x', z, t) &= \int_{-a}^a A(t - \tau(x, x', z) + \tau'(x, x', z)) \cos \omega_0(t - \tau(x, x', z) + \tau'(x, x', z)) dx \\ &= \int_{-a}^a A\left(t - \frac{z}{c} + \frac{xx'}{zc} - \frac{x'^2}{2zc}\right) \cos \omega_0\left(t - \frac{z}{c} + \frac{xx'}{zc} - \frac{x'^2}{2zc}\right) dx \end{aligned}$$

Based on the above equation, we find that at $x' = 0$

$$p(0, z, t) = \int_{-a}^a A\left(t - \frac{z}{c}\right) \cos \omega_0\left(t - \frac{z}{c}\right) dx = 2aA\left(t - \frac{z}{c}\right) \cos \omega_0\left(t - \frac{z}{c}\right).$$

In other words, $p(0, z, t)$ is simply the replica of the original pulse delayed by

z/c and multiplied by the size of the aperture. It is also implied that at the focal point, signals coming from the aperture should be constructively summed.

When $x' \neq 0$ (i.e., off-axis), the pressure becomes

$$p(x', z, t) = \text{Re} \left\{ \int_{-a}^a A \left(t - \frac{z}{c} + \frac{xx'}{zc} - \frac{x'^2}{2zc} \right) e^{j\omega_0 \left(t - \frac{z}{c} + \frac{xx'}{zc} - \frac{x'^2}{2zc} \right)} dx \right\}$$

$$= \text{Re} \left\{ e^{j\omega_0 \left(t - \frac{z}{c} - \frac{x'^2}{2zc} \right)} \int_{-a}^a A \left(t - \frac{z}{c} + \frac{xx'}{zc} - \frac{x'^2}{2zc} \right) e^{j\omega_0 \frac{xx'}{zc}} dx \right\}.$$

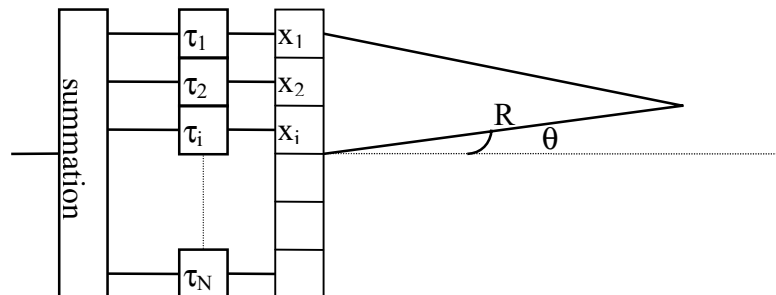
The above equation is simply the same diffraction equation weighted by an envelope term. In the limit where $\frac{xx'}{zc} - \frac{x'^2}{2zc}$ is small (i.e., in the vicinity of $x' = 0$), we obtain

$$p(x', z, t) = \text{Re} \left\{ e^{j\omega_0 \left(t - \frac{z}{c} - \frac{x'^2}{2zc} \right)} A \left(t - \frac{z}{c} \right) \int_{-a}^a e^{j\omega_0 \frac{xx'}{zc}} dx \right\},$$

i.e., the radiation pattern can be approximated by the product of the excitation pulse and the continuous wave diffraction pattern if it is near the focal point.

IV. Beam Formation Using Arrays

- In the previous section, we have shown that a lens can be formed by using time delays across the aperture. It also becomes apparent that how an array can be used to form a beam focused at any particular point in space. Since the reception is simply reciprocal to transmission, a received beam can also be formed at any particular point in space by properly applying time delays to the received signals prior to the summation across the aperture. This is why beam formation is usually referred to as “delay-and-sum”. The receive beam formation can be illustrated by the following drawing



x_N

- Since an array can focus beams at an off-axis point, it can replace the need for mechanical scanning and real-time images can be acquired through electronic focusing and steering.
- By using a sector scan format, any image point in space can be specified by polar coordinates (R, θ) . Assuming $S_i(t)$ is the received signal on the i -th element, the diffraction integral shown in the previous section can be approximated by the following coherent summation

$$O(t) = \sum_{i=1}^N S_i(t - \tau(x_i, R, \theta)),$$

where $O(t)$ is the output signal and

$$\tau(x_i, R, \theta) = \frac{\left((x_i - R \sin \theta)^2 + R^2 \cos^2 \theta \right)^{1/2}}{c} = \frac{R}{c} \left(1 + \frac{x_i^2}{R^2} - \frac{2x_i}{R} \sin \theta \right)^{1/2}.$$

In the Fresnel region,

$$\begin{aligned} \tau(x_i, R, \theta) &\approx \frac{R}{c} \left(1 + \frac{x_i^2}{2R^2} - \frac{x_i}{R} \sin \theta - \frac{x_i^2}{2R^2} \sin^2 \theta \right) \\ &= \frac{R}{c} \left(1 - \frac{x_i}{R} \sin \theta + \frac{x_i^2}{2R^2} \cos^2 \theta \right) = \frac{R}{c} - \frac{x_i \sin \theta}{c} + \frac{x_i^2 \cos^2 \theta}{2Rc}. \end{aligned}$$

- The first term in the above equation is independent of the angle and the channel index, it is simply the propagation time from the center of the array to a range R . The second term relates to the beam direction and is independent of the range. It represents the steering component. The third term is a parabolic function and represents the focusing component. It approaches to zero in the far field.
- The third term in the above equation also implies that when the focal point is off-axis ($\theta \neq 0$), the effective aperture size also reduces from $2a$ to $2a \cos \theta$. In other words, the lateral resolution is reduced when the scan angle is steered off normal.

- On transmit, since the array is only fired once, a fixed focus must be specified for the transmit beam (for that particular firing). On receive, on the other hand, since the received signal is continuously stored as a function of time, the receive focus can be continuously updated. Therefore, the receive can be dynamically focused in the image range.
- Based on the previous discussion, the transmit beam is formed by applying relative time delays (relative to the center of the array) across the transducer, i.e.,

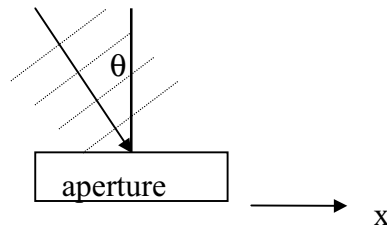
$$\tau^T(x_i, R, \theta) = -\frac{x_i \sin \theta}{c} + \frac{x_i^2 \cos^2 \theta}{2Rc},$$

and the receive delay is

$$\tau^R(x_i, R, \theta) = \frac{2R}{c} - \frac{x_i \sin \theta}{c} + \frac{x_i^2 \cos^2 \theta}{2Rc}.$$

V. More on the Radiation Pattern Using Arrays

- Previous diffraction equations can be re-written by replacing x'/z with $\sin \theta$. Consider the following one-dimensional aperture and a continuous plane wave (i.e., a wave from the far field), the signal received by the aperture as a function of the wave direction is

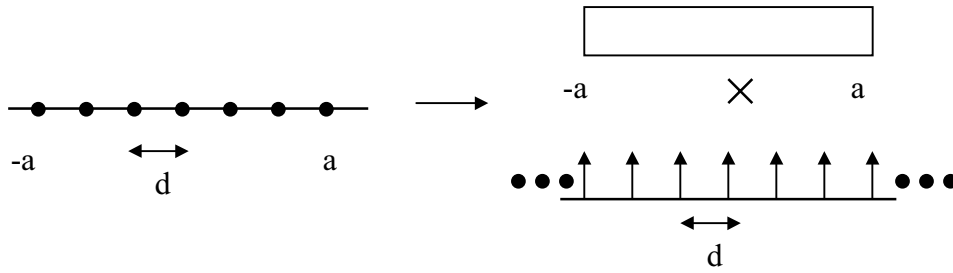


$$p(\theta) = \int_{-a}^a C(x) e^{-jkx \sin \theta} dx.$$

It is obvious that the Fourier transform relation still holds.

- The above equation assumes a continuous aperture. Considering a sampled

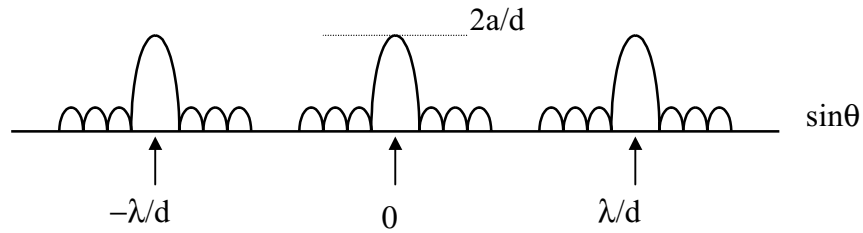
aperture (i.e., an array) as the following,



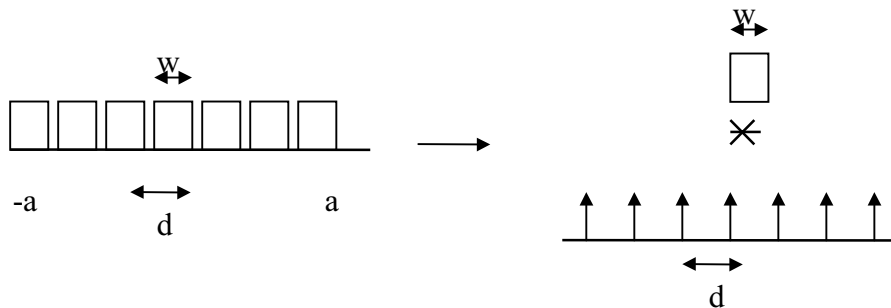
the aperture function can be viewed as the original continuous aperture function (rectangular) multiplied by an “impulse train” with a spacing d . Using the Fourier transform pair (defining $f = \sin \theta / \lambda$)

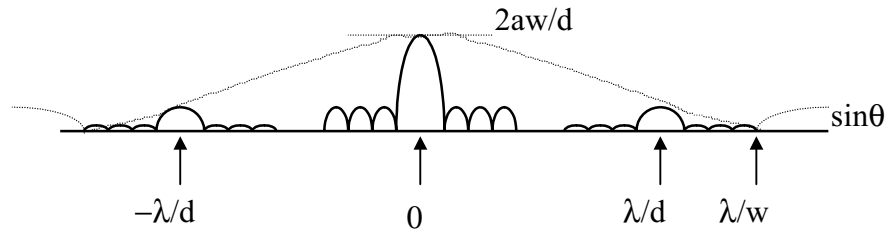
$$\sum_{i=-\infty}^{\infty} \delta(x - id) \leftrightarrow \frac{1}{d} \sum_{i=-\infty}^{\infty} \delta(f - i/d),$$

the new radiation pattern is the convolution of the f domain impulse train and the original radiation pattern for the continuous aperture, i.e.,

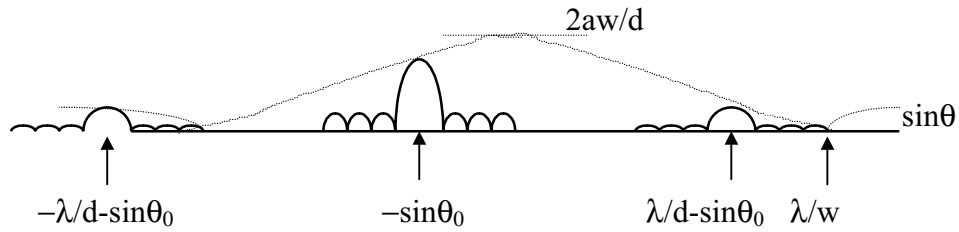


- Practically, each array element has a finite width w ($w < d$). Hence, the array function becomes an “impulse train” convolves with a rectangular function. The resultant radiation pattern is then the multiplication of the point array and the radiation pattern of a single transducer element.

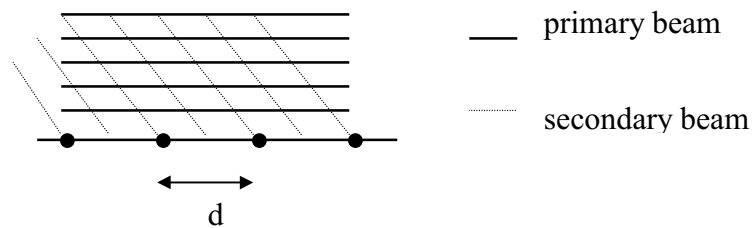




- When a beam is steered to a direction θ_0 , linear time delays are applied and therefore the radiation pattern is shifted towards the steering angle. However, the radiation pattern from a single transducer element is not shifted. Therefore,



- Note that as the main beam is steered, secondary beams become stronger. These secondary beams are also called grating lobes and are due to the fact that when the spacing between adjacent elements is sufficiently large, there are several directions of the source that can be summed in phase, just as the signals from the main source. It can also be viewed as aliases in space.



- The horizontal axis in the above radiation pattern drawing is $\sin \theta$. Physically, $\sin \theta$ has to be between -1 and 1. Therefore, to prevent grating lobes, we can have

$$2 \leq \lambda / d ,$$

or

$$d \leq \lambda / 2 .$$

The criterion is equivalent to the Nyquist criterion for temporal sampling.

VI. Spatial Sampling Criteria

- Another way to look at grating lobes is based on the fact that spatial aliasing occurs when the phase shift between adjacent transducer elements is larger than π . In other words,

$$2\pi f_0 (\tau(x_i, R, \theta) - \tau(x_{i+1}, R, \theta)) > \pi.$$

Using the equations derived previously and assuming R is sufficiently large, we have

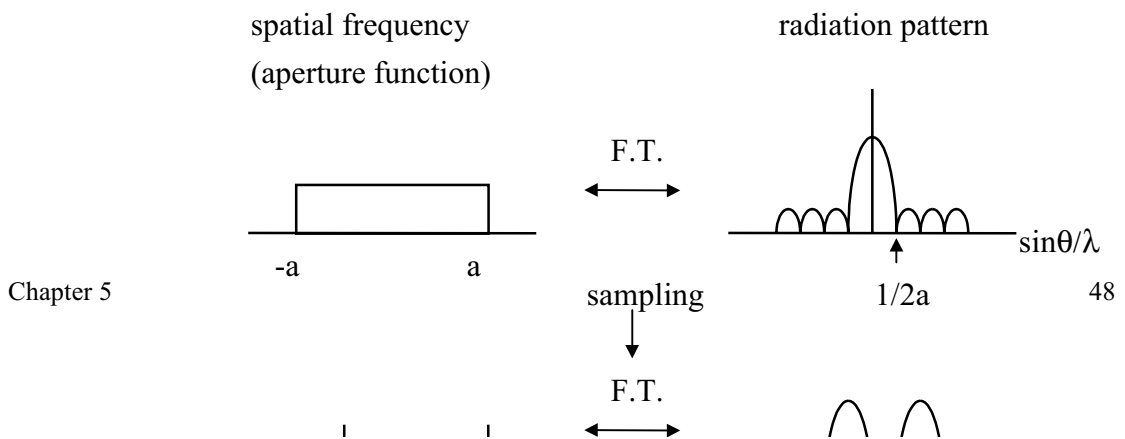
$$\begin{aligned} & 2\pi f_0 \left(\frac{R}{c} - \frac{x_i \sin \theta}{c} - \frac{R}{c} + \frac{x_{i+1} \sin \theta}{c} \right) \\ &= \frac{2\pi f_0 \sin \theta}{c} (x_{i+1} - x_i) = \frac{2\pi \sin \theta d}{\lambda} > \pi \end{aligned}$$

Therefore, grating lobes occur when

$$\sin \theta > \frac{\lambda}{2d}.$$

Since $|\sin \theta| \leq 1$, we know that grating lobes will not occur when $d \leq \lambda/2$. This is consistent with our previous conclusion.

- The physical meaning of no grating lobes is that the beam can be steered unambiguously. In other words, the beam can be steered to any direction in space and the received signals are coming only from the main beam direction.
- As described previously, a two-dimensional image consists of a finite set of one-dimensional lines (A-scans). In other words, the two-dimensional image is also sampled and the spacing between beams also needs to be close enough to avoid aliasing. This beam spacing can be derived using Nyquist criterion in the spatial frequency domain shown below



To satisfy the spatial Nyquist criterion (letting the beam spacing be $\Delta \sin \theta$), we have

$$\frac{1}{\Delta \sin \theta / \lambda} \equiv \frac{1}{b} \geq 2a.$$

Therefore, $\Delta \sin \theta \leq \frac{\lambda}{2a}$. In other words, the spacing should be less than the location of the first zero-crossing point. For pulse-echo imaging, the two-way effective aperture is the convolution of the transmit aperture with the receive aperture (since the two-way radiation pattern is the product of the transmit pattern and the receive pattern), the two-way effective aperture size becomes $4a$ and the beam spacing should be $\Delta \sin \theta \leq \frac{\lambda}{4a}$.

- If the beam spacing is less than the Nyquist criterion, no additional information will be obtained and the frame rate will be reduced since there are more ultrasound lines need to be acquired. This is equivalent to the temporal sampling theorem.
- Again, the spatial Nyquist criterion can be obtained by looking at the phase across the aperture. With a beam spacing $\Delta \sin \theta$ satisfying the Nyquist criterion ($\Delta \sin \theta = \lambda / 2a$), the phase difference across the aperture between two adjacent beams becomes (ignoring range R)

$$\begin{aligned} & 2\pi f_0 (\tau(a, R, \theta_{i+1}) - \tau(-a, R, \theta_{i+1}) - (\tau(a, R, \theta_i) - \tau(-a, R, \theta_i))) \\ &= \frac{2\pi f_0}{c} (2a \Delta \sin \theta) = \frac{2\pi f_0}{c} \frac{2a\lambda}{2a} = 2\pi \end{aligned}$$

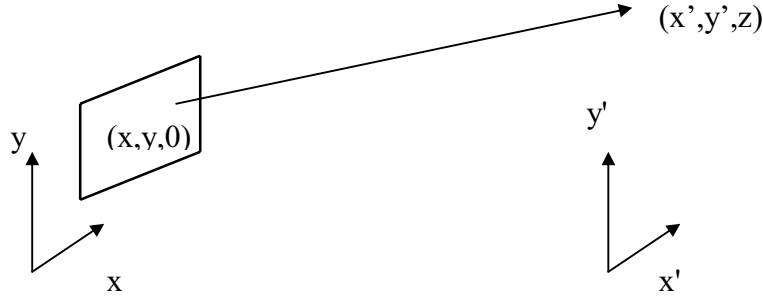
In other words, the spatial Nyquist criterion states that when the phase difference across the aperture between two adjacent beams is larger than 2π (one cycle), aliasing occurs.

VII. Diffraction for Two-Dimensional Apertures

- The derivation for diffraction from two-dimensional apertures is similar to one-dimensional situations. For any point in the three-dimensional space, the pressure wave is the integral of waves from all point radiators on the aperture and each point radiator has its own amplitude and phase (or delay) relative to the point in space. In other words,

$$p(x', y', z) = \iint_{\text{area}} \frac{e^{jkd((x, y), (x', y'))}}{d((x, y), (x', y'))} dx dy,$$

where $d((x, y), (x', y'))$ is the distance between a point (x, y) in the aperture plane (plane $z = 0$) to a point (x', y') in plane z .



- In the Fresnel region (or far field), the two-dimensional radiation pattern can also be obtained by Fourier transforming the two-dimensional effective aperture function. If the aperture function is separable, i.e.

$$C(x, y) = C(x)C(y),$$

the radiation pattern is also separable, i.e.,

$$B(x', y', z) = B(x', z) \cdot B(y', z) = F.T.[C(x)] \cdot F.T.[C(y)].$$

Note that since a rectangular function is separable, the radiation pattern from a rectangular aperture is the product of two sinc functions.

- For two-dimensional circular apertures, the radiation patterns can be found by

using one-dimensional transforms based on symmetry. Assuming a uniformly apodized circular aperture, the radiation pattern also resembles a sinc function, except that the sine function in the numerator is replaced by the first-order Bessel function. Such a function is also known as a “jinc” function.

- For real-time two-dimensional imaging, one-dimensional arrays are usually adequate. In other words, the scanning in x is done electronically, as we have discussed previously and there is no scanning in y . To provide better focusing quality in the main region of interest of a particular transducer, there is usually a mechanical lens attached to the front of the array in order to provide fixed, geometric focusing in the non-scan direction (i.e., y). To perform real-time three-dimensional imaging electronically, on the other hand, fully sampled (Nyquist criterion is satisfied in both x and y) two-dimensional arrays are obviously required.